Gravitational waves from first-order phase transitions

Lecture 1: phase transitions and thermal field theory

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Relevant references for these lectures

- The course these three lectures follow: Laine, *Phase transitions in the early universe*, 2022, `laine.itp.unibe.ch/ggi22.pdf`

- Thermal field theory from the ground up: Laine, Vuorinen, *Basics of Thermal Field Theory*, vol. 925 of *Lecture Notes in Physics*. Springer International Publishing, Cham, Jan, 2016, [1701.01554]

- Transitions, hydrodynamics and the signal: Hindmarsh, Lüben, Lumma, Pauly, *Phase transitions in the early universe*, SciPost Phys. Lect. Notes **24** (2021) 1 [2008.09136]

- What LISA can expect to see: Caprini, Chala, Dorsch *et al.*, *Detecting gravitational waves from cosmological phase transitions with LISA: an update*, JCAP **03** (2020) 024 [1910.13125]

Plan of the three lectures

1. **Phase transitions and thermal field theory**
   What is a phase transition? Thermal field theory, the effective potential, and the infrared problem.
2. **The 3d EFT, and nucleation**
   Dimensional reduction, the Standard Model and beyond; then the bounce, supercooling and the nucleation rate.
3. **Wall velocity and spectra**
   Hydrodynamics of the expanding bubble, friction and the Boltzmann equation, v_w, and the resulting gravitational-wave spectrum.

Equilibrium spans lectures 1 and 2; real time starts with nucleation.

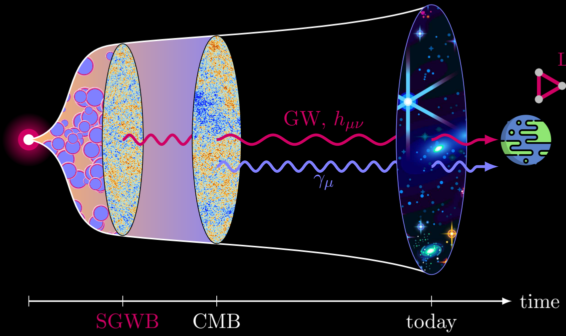
Goal: connect the microscopic Lagrangian of a model to a macroscopic gravitational-wave signal.

Motivation

Gravitational waves can be sourced by the early universe

A first-order phase transition caused by new physics leaves a stochastic gravitational-wave background (SGWB).

Gravitons free-stream from $t \sim 10^{-11}$ s (electroweak scale) until today. Photons only do so from the CMB, at $t \simeq 3.8 \times 10^5$ yr.



Cosmic Microwave Background (CMB)
Stochastic GW Background (SGWB)
Laser Interferometer Space Antenna (LISA)

The Laser Interferometer Space Antenna (LISA)

A GW detector in space, flown by ESA:

- three spacecraft in an equilateral triangle, arms of 2.5×10^6 km, trailing the Earth on its orbit;
- 2024: adopted by ESA;
- 2035: launch on Ariane 6, nominal mission ~ 4 years.

The arm length sets the band: $c/(2\pi L) \simeq 19$ mHz, most sensitive for 10^{-4}–10^{-1} Hz.

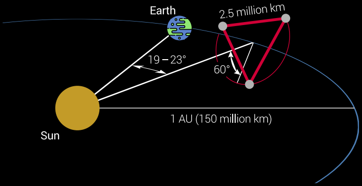
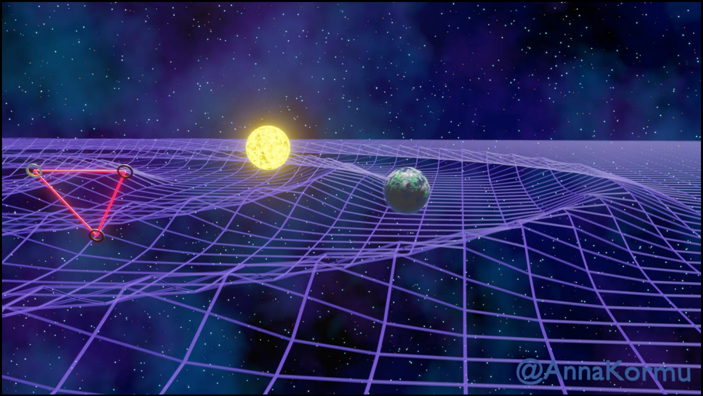


figure by **LISA** Collaboration, Amaro-Seoane *et al.*, *Laser Interferometer Space Antenna*, [1702.00786]

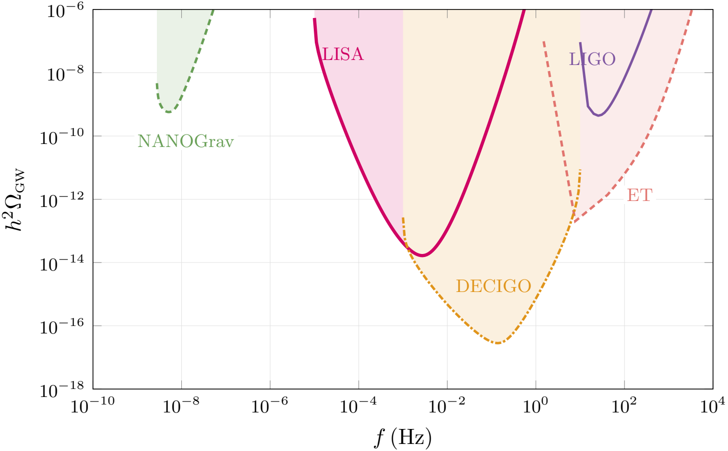
Three spacecraft, one interferometer



A passing wave stretches one arm and squeezes another, by $\delta L/L \sim h \sim 10^{-21}$ over 2.5 million km.

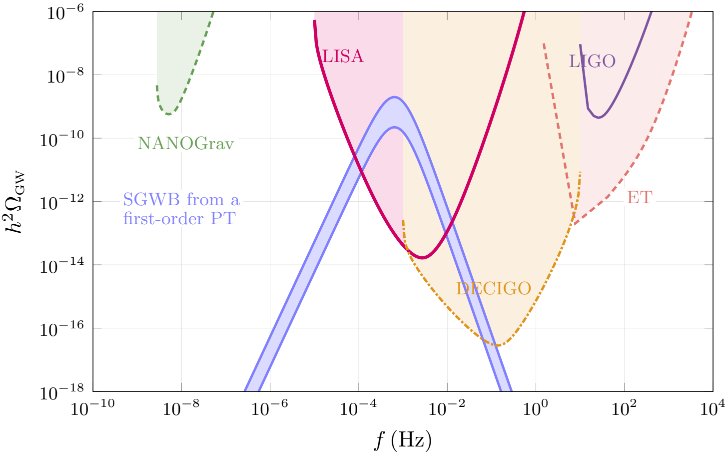
visualisation by Anna Kormu, blogs.helsinki.fi/koranna/visualisations

A new window on the early universe



Each detector covers its own band: pulsar timing at nHz, LISA at mHz, LIGO and ET at 10–10^3 Hz.

A new window on the early universe



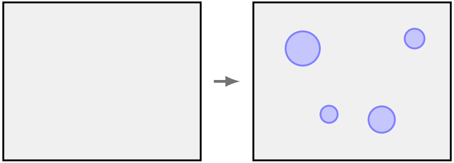
A first-order transition at $T \sim 100$ GeV gives a *stochastic* background peaked at $f \sim$ mHz today: inside the LISA band.

What a first-order transition does

cooled below T_c:
still the old phase

Below T_c the old phase does not convert at once: it is metastable, and the universe *supercools*.

What a first-order transition does



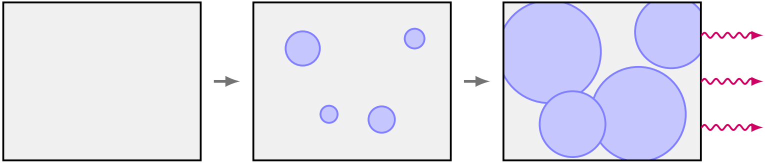
cooled below T_c:
still the old phase

bubbles nucleate
and grow

Below T_c the old phase does not convert at once: it is metastable, and the universe *supercools*.

Bubbles of the new phase appear at random points and grow, pushed by the free-energy difference between the two phases.

What a first-order transition does



cooled below T_{c}: still the old phase

bubbles nucleate and grow

they collide: sound waves radiate

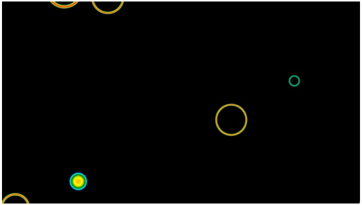
Below T_{c} the old phase does not convert at once: it is metastable, and the universe *supercools*.

Bubbles of the new phase appear at random points and grow, pushed by the free-energy difference between the two phases.

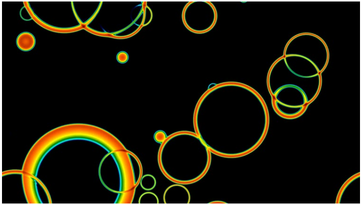
Colliding bubbles set off sound waves, whose anisotropic stress $T_{ij} \supset w\gamma^2 v_i v_j$ radiates: $\Box\, \bar{h}_{ij} = -16\pi G\, T_{ij}$.

And this is what it looks like

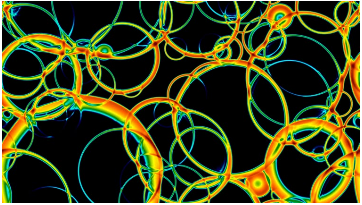
Stills from a simulation of the plasma; each stage is one of the three lectures.



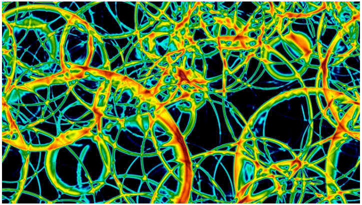
bubbles nucleate



and grow



they collide

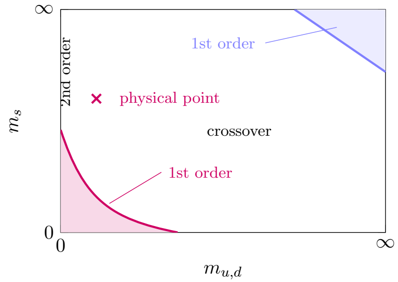


and sound waves remain

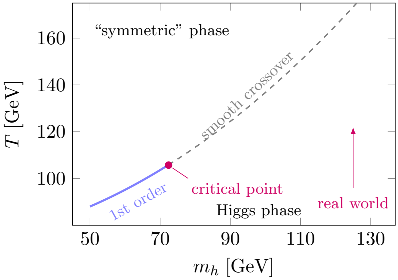
[1] Simulation and video by D. Weir, `saoghal.net/slides/brda`; full movie `youtu.be/mfGL8Cp0RPA`

Both symmetries are restored, but smoothly

Both QCD and the electroweak theory display a broken symmetry in vacuum, which gets *smoothly restored* at high temperatures.



QCD: the quark masses[1]

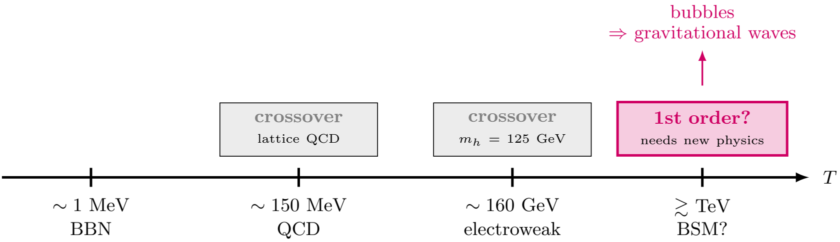


electroweak: the Higgs mass

Both physical points fall in the crossover region: nothing boils, nothing radiates. A first-order transition needs new physics.

[1] Aoki, Endrődi, Fodor, Katz, Szabó, *The order of the quantum chromodynamics transition predicted by the standard model of particle physics*, Nature **443** (2006) 675 [0611014]

A thermal history with (possibly) several transitions



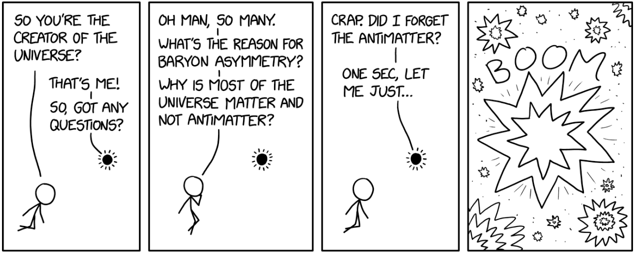
In the Standard Model both the QCD ($T \simeq 155$ MeV) and the electroweak ($T \simeq 160$ GeV) transition are the *crossovers* of the previous slide: no bubbles, no gravitational waves.

A first-order transition therefore requires physics beyond the Standard Model, e.g. new scalars coupled to the Higgs.

And a second reason: where did the antimatter go?

The universe is made of matter, with essentially no antimatter left: about one baryon per billion photons survived annihilation, as nucleosynthesis and the CMB both measure:[1]

$$\eta_B \equiv \frac{n_B - n_{\bar{B}}}{n_\gamma} \simeq 6.1 \times 10^{-10}\,, \qquad \frac{n_B}{s} = (8.72 \pm 0.05) \times 10^{-11}\,.$$



xkcd 3263, Randall Munroe, CC BY-NC 2.5

[1] **Planck** Collaboration, Aghanim *et al.*, *Planck 2018 results. VI. Cosmological parameters,* Astron. Astrophys. **641** (2020) A6 [1807.06209]

Three conditions, and the Standard Model fails one

By hand? Inflation dilutes it away.
In equilibrium? $\eta_B \equiv 0$.
It must be generated *dynamically*, which takes three things:[1]

Sakharov monument, Yerevan.

Photo: Beko, CC BY-SA 4.0

[1] Sakharov, *Violation of CP Invariance, C asymmetry, and baryon asymmetry of the universe*, Pisma Zh. Eksp. Teor. Fiz. **5** (1967) 32

Three conditions, and the Standard Model fails one

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- ✓ B violation: sphalerons, active above $T \simeq 130$ GeV;
- ✓ C and CP violation: only the CKM phase, suppressed to $\sim 10^{-20}$;

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Photo: Beko, CC BY-SA 4.0

[1] Sakharov, *Violation of CP Invariance, C asymmetry, and baryon asymmetry of the universe,* Pisma Zh. Eksp. Teor. Fiz. **5** (1967) 32

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> Condition 3) is exactly a first-order phase transition: the same object that sources gravitational waves.

Sakharov monument, Yerevan.

Photo: Beko, CC BY-SA 4.0

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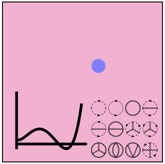
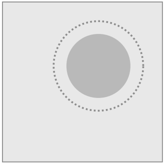
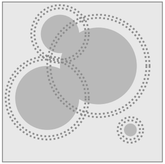
From a Lagrangian to a gravitational-wave signal



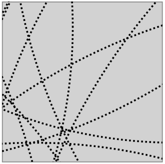
a Lagrangian



one bubble



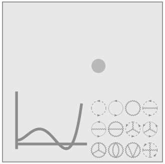
many bubbles



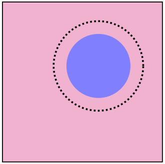
sound waves

(a) the effective potential of a given model (**today**);
(b)
(c)

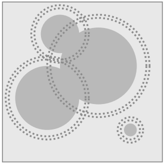
From a Lagrangian to a gravitational-wave signal



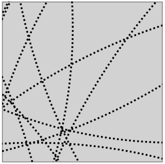
a Lagrangian



one bubble



many bubbles



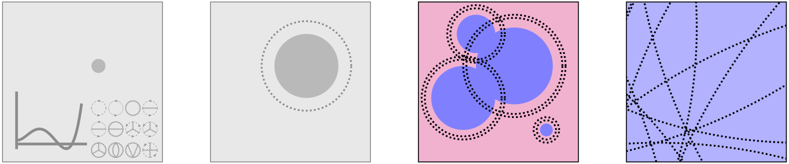
sound waves

(a) the effective potential of a given model (**today**);

(b) which bubbles nucleate, and when (**lecture 2**);

(c)

From a Lagrangian to a gravitational-wave signal



a Lagrangian | one bubble | many bubbles | sound waves

(a) the effective potential of a given model (**today**);

(b) which bubbles nucleate, and when (**lecture 2**);

(c) how fast they grow, and what the fluid does when they collide (**lecture 3**).

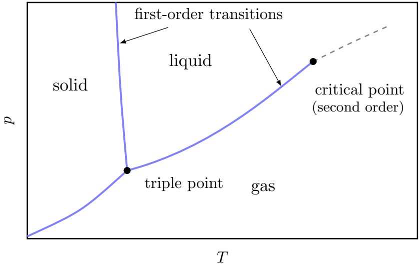
Everything funnels through four numbers: T_n, α, $\beta/H_\star$ and v_w.

What is a phase transition?

A phase transition: non-analyticity of the free energy

A phase transition is a point where the free energy $f(T, p, \ldots)$ is *non-analytic*:

- first order: f continuous, $\partial f / \partial T$ jumps $\Rightarrow$ latent heat L;
- second order: $\partial f / \partial T$ continuous, $\partial^2 f / \partial T^2$ jumps;
- crossover: f analytic, no transition.



Water: a first-order line ending in a critical point. The electroweak phase diagram has the same shape, with an endpoint at $m_h \simeq 72$ GeV.

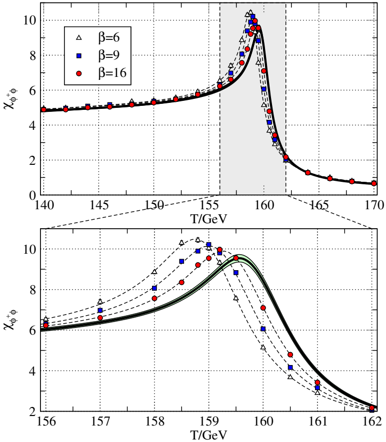
Thermodynamics. At $\mu = 0$ the densities obey

$$f = -p\,, \qquad s = -\frac{\partial f}{\partial T}\,.$$

Latent heat. f is continuous across the transition, its slope is not:

$$L \equiv \Delta e = -T_\mathrm{c}\,\Delta\Big(\frac{\mathrm{d}f}{\mathrm{d}T}\Big) \neq 0\,.$$

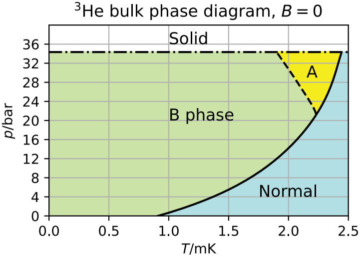
Field theory.
order parameter: $\langle\phi\rangle$;
free energy: $V_\mathrm{eff}(\phi, T)$,
the effective potential.

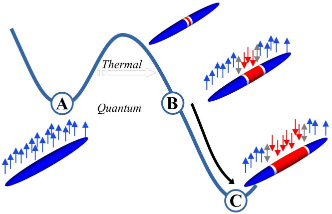


Higgs susceptibility $\chi_{\phi^\dagger\phi}$ across the electroweak crossover

The same happens in condensed-matter systems

Two laboratory analogues of bubble nucleation





Superfluid ^{3}He: the A–B transition is seen, yet the predicted lifetime exceeds the age of the universe.[1]

Cold atoms (1+1d ferromagnetic superfluid): exponent of the rate confirmed, prefactor fitted.[2]

[1] **QUEST-DMC** Collaboration, Hindmarsh *et al.*, *A-B Transition in Superfluid ^{3}He and Cosmological Phase Transitions*, J. Low Temp. Phys. **215** (2024) 495 [2401.07878]

[2] Zenesini, Berti, Cominotti *et al.*, *False vacuum decay via bubble formation in ferromagnetic superfluids*, Nature Phys. **20** (2024) 558 [2305.05225]; Cominotti, Baroni, Rogora *et al.*, *Observation of Temperature Effects on False Vacuum Decay in Atomic Quantum Gases*, Phys. Rev. Lett. **135** (2025) 183401 [2504.03528]

Thermal field theory in a nutshell

From one oscillator to a field theory

Everything below follows from one spectrum, $\epsilon \equiv \hbar\omega$, at temperature $T \equiv 1/\beta$ ($k_{\mathrm{B}} = 1$):

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} = \hbar\omega\Big(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\Big)\,, \qquad \hat{H}\,|n\rangle = \epsilon\Big(n + \tfrac{1}{2}\Big)|n\rangle\,.$$

1. The partition function. Take the trace in the occupation-number basis. This is a geometric series, convergent because $e^{-\beta\epsilon} < 1$:

$$\mathcal{Z} = \mathrm{Tr}\, e^{-\beta\hat{H}} = \sum_{n=0}^{\infty}\langle n|e^{-\beta\hat{H}}|n\rangle = e^{-\beta\epsilon/2}\sum_{n=0}^{\infty}\big(e^{-\beta\epsilon}\big)^n = \frac{e^{-\beta\epsilon/2}}{1 - e^{-\beta\epsilon}}\,.$$

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2. The free energy is its logarithm, and splits into a temperature-independent zero-point piece and a thermal one:

$$F = -T\ln\mathcal{Z} = \underbrace{\frac{\epsilon}{2}}_{\text{vacuum}} + \underbrace{T\ln\big(1 - e^{-\beta\epsilon}\big)}_{\text{thermal}}\,.$$

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3. The propagator needs no new work: $\hat{x}^2$ is inside $\hat{H}$ multiplied by $m\omega^2/2$, so differentiating with respect to ω^2 brings it down from the exponent,

$$\langle \hat{x}^2 \rangle = \frac{\mathrm{Tr}\big(\hat{x}^2 e^{-\beta\hat{H}}\big)}{\mathrm{Tr}\, e^{-\beta\hat{H}}} = \frac{2}{m}\,\frac{\partial F}{\partial \omega^2} = \frac{\hbar^2}{m}\frac{1}{\epsilon}\frac{\partial F}{\partial \epsilon}\,,$$

where the last step used $\epsilon = \hbar\omega$, so $\partial\epsilon/\partial\omega^2 = \hbar^2/(2\epsilon)$.

From one oscillator to a field theory

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4. Carry out the derivative of the F from step 2:

$$\langle\hat{x}^2\rangle = \frac{\hbar^2}{m}\frac{1}{\epsilon}\Bigg[\underbrace{\;\frac{1}{2}\;}_{\text{vacuum}} + \underbrace{\frac{e^{-\beta\epsilon}}{1-e^{-\beta\epsilon}} = \frac{1}{e^{\beta\epsilon}-1}}_{\text{Bose}}\Bigg]\,.$$

The vacuum half survives at $T = 0$; the Bose half is the whole of the thermal physics.

From one oscillator to a field theory

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5. Promote it to a field. A free scalar field is one oscillator per momentum mode, so $\epsilon \to \epsilon_k$ and the modes are summed over:

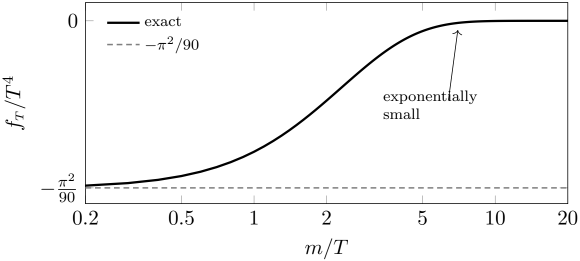
$$f_\phi = \underbrace{\int_{\mathbf{k}} \frac{\epsilon_k}{2}}_{\text{vacuum}} + \underbrace{T\int_{\mathbf{k}} \ln\big(1 - e^{-\beta\epsilon_k}\big)}_{\equiv\ f_T(m),\ \text{thermal}}\,, \qquad \epsilon_k = \sqrt{k^2 + m^2}\,.$$

> Bose enhancement: for $\epsilon_k \ll T$ the occupation is $n_{\rm B}(\epsilon_k) \simeq T/\epsilon_k \gg 1$. This drives the infrared problem at the end of this lecture.

The high-temperature expansion is non-analytic

$$f_T(m) = -\frac{\pi^2 T^4}{90}$$

- leading term: Stefan–Boltzmann;

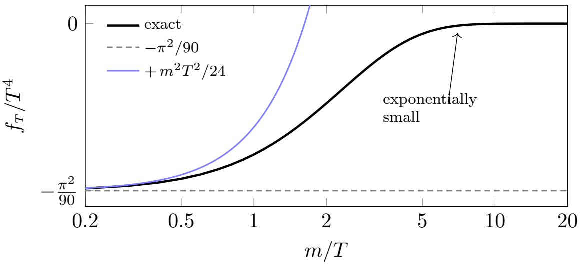


Each term as it is added; past $m \sim T$ they run away, while $f_T \approx -(mT/2\pi)^{3/2} e^{-m/T}$: heavy modes decouple.

The high-temperature expansion is non-analytic

$$f_T(m) = -\frac{\pi^2 T^4}{90} + \frac{m^2 T^2}{24}$$

- leading term: Stefan–Boltzmann;
- m^2T^2: thermal mass, symmetry restoration;



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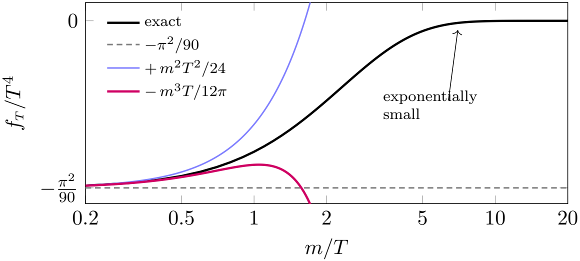
The high-temperature expansion is non-analytic

$$f_T(m) = -\frac{\pi^2 T^4}{90} + \frac{m^2 T^2}{24} - \frac{m^3 T}{12\pi} + \mathcal{O}(m^4)\,.$$

- leading term: Stefan–Boltzmann;
- m^2T^2: thermal mass, symmetry restoration;
- m^3T: odd, non-analytic, from Matsubara *zero mode*.



Each term as it is added; past $m \sim T$ they run away, while $f_T \approx -(mT/2\pi)^{3/2} e^{-m/T}$: heavy modes decouple.

Matsubara frequencies and the zero mode

The propagator can be written as a sum over discrete frequencies,

$$\frac{1}{\epsilon_k}\left[\frac{1}{2}+n_{\mathrm{B}}(\epsilon_k)\right] \;=\; T\sum_{\omega_n}\frac{1}{\omega_n^2+\epsilon_k^2}\,, \qquad \omega_n = 2\pi T n\,, \quad n\in\mathbb{Z}\,.$$

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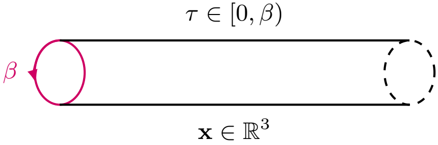
Remember this: all $n \neq 0$ modes carry a hard mass $\geq 2\pi T$ and are therefore infrared safe. Only $\omega_n = 0$ is dangerous. And $\omega_n = 0$ means: no τ dependence.

Imaginary time: why the zero mode is three-dimensional

$$\mathcal{Z}_\phi = \int_{\text{b.c.}} \mathcal{D}\phi \, e^{-S_{\text{E}}} \,, \qquad\qquad S_{\text{E}} = \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \; \mathcal{L}_{\text{E}} \,,$$

with $\mathcal{L}_{\text{E}} = -\mathcal{L}_{\text{M}}(t \to -\mathrm{i}\tau)$: τ runs around a circle of circumference $\beta = 1/T$, and periodicity is what quantises $\omega_n = 2\pi T n$.



Expand along the circle, $\phi(\tau, \mathbf{x}) = T \sum_n \phi_n(\mathbf{x}) \, e^{\mathrm{i}\omega_n \tau}$: the $n = 0$ term is the one that does *not* wind around it.

> So ϕ_0 is a field on $\mathbb{R}^3$ alone: the one dangerous mode lives in three dimensions. As T grows the circle shrinks away and a classical 3d theory is left.

The effective potential

Split $\phi = \bar{\phi} + \phi'$ into the constant mode and the rest, and integrate out ϕ':

$$\mathcal{Z}_\phi = \int \mathrm{d}\bar{\phi}\; e^{-\frac{V}{T} V_{\mathrm{eff}}(\bar{\phi})} \quad \Longrightarrow \quad f_\phi = V_{\mathrm{eff}}(\bar{\phi}_{\mathrm{min}}) + \mathcal{O}\Big(\frac{\ln V}{V}\Big)\,.$$

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Order by order,

$$\begin{aligned} V_{\text{eff}}^{(0)} &= -\frac{m^2\bar{\phi}^2}{2} + \frac{\lambda\bar{\phi}^4}{4}\,, \\ V_{\text{eff}}^{(1)} &= \int_{\mathbf{k}} \Big[\frac{\epsilon_k}{2} + T\ln\big(1 - e^{-\beta\epsilon_k}\big)\Big]\bigg|_{\epsilon_k = \sqrt{k^2 + m_{\text{eff}}^2}}\,, \end{aligned}$$

with $m_{\text{eff}}^2 = -m^2 + 3\lambda\bar{\phi}^2$.

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with $m_{\text{eff}}^2 = -m^2 + 3\lambda\bar{\phi}^2$.

Two problems already visible: ϵ_k is complex for $\bar{\phi}^2 < m^2/3\lambda$, and $V_{\text{eff}}^{(0)} \sim V_{\text{eff}}^{(1)} \sim V_{\text{eff}}^{(2)}$ once $m_{\text{eff}} \lesssim \lambda T/\pi$.

Symmetry restoration, and an apparent barrier

Insert the high-T expansion. The m^2T^2 term restores the symmetry,

$$V_{\rm eff} \simeq \frac{1}{2}\Big(-m^2 + \frac{\lambda T^2}{4}\Big)\bar{\phi}^2 + \frac{\lambda}{4}\bar{\phi}^4 \;\Rightarrow\; T_{\rm c} = \frac{2m}{\sqrt{\lambda}}\,,$$

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while the m^3T term (at $m^2 = 0$) generates a barrier,

$$V_{\text{eff}} \simeq \frac{\lambda T^2}{8}\bar{\phi}^2 - \frac{T(3\lambda)^{3/2}}{12\pi}|\bar{\phi}|^3 + \frac{\lambda}{4}\bar{\phi}^4\,,$$

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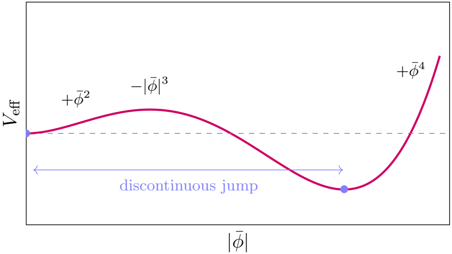
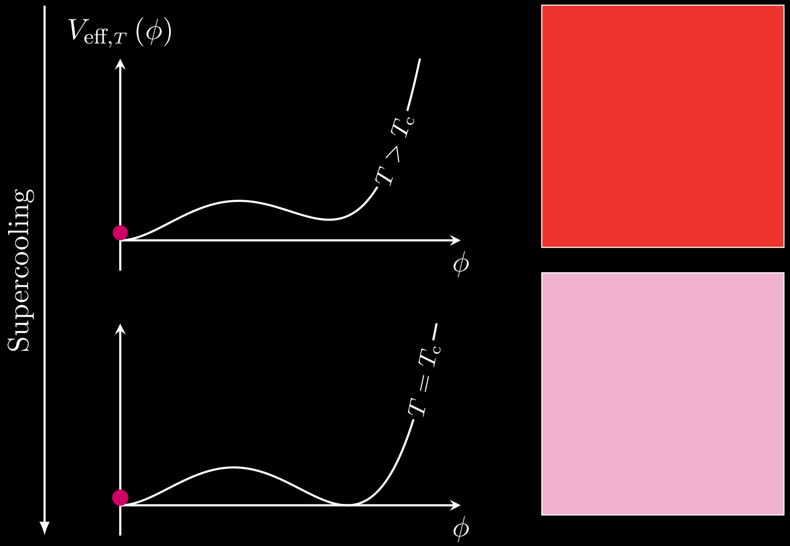
and with it a jump of the order parameter to $\bar{\phi}_{\mathrm{c}}/T_{\mathrm{c}} = \sqrt{3\lambda}/(2\pi)$: this *looks* first order.



Cosmological first order phase transition



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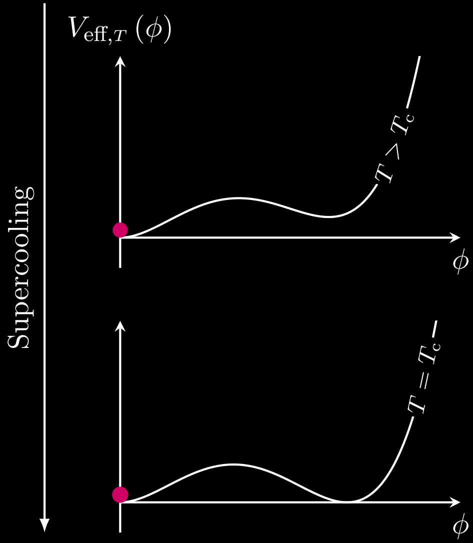
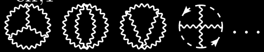
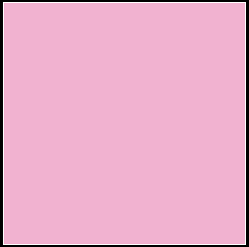


UV [thermal] corrections modify the zero-T potential and restore symmetry.

$V_{\text{eff},T} \supset$





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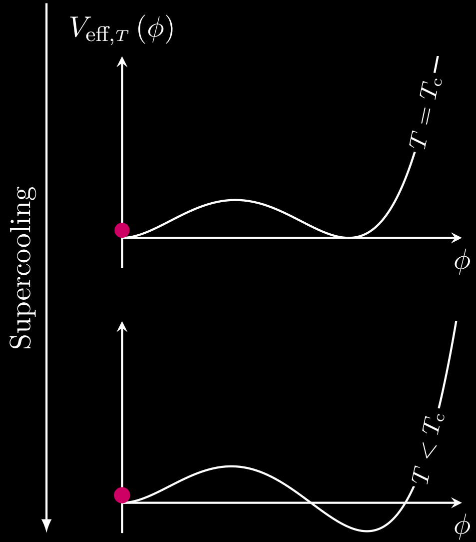
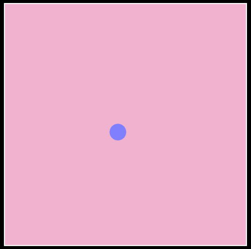


Below T_c the broken minimum is the lower one.

If a barrier separates the phases, the transition is first order.



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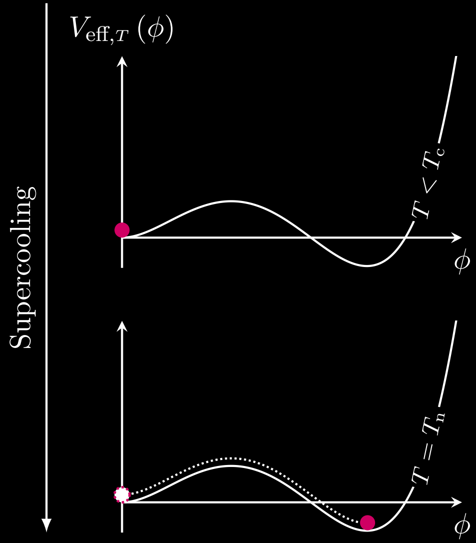
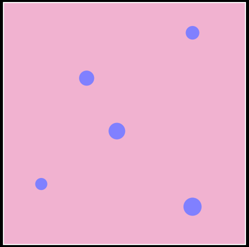


The phase transition proceeds via nucleation of bubbles.

Part of the released energy can be converted into gravitational waves: $\Box\,\bar{h}_{ij} = -16\pi G\,T_{ij}$



Problems

Problem 1: Verify the Matsubara sum rule by evaluating

$$\oint \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{\mathrm{i}\, n_{\mathrm{B}}(\mathrm{i}\omega)}{\omega^2 + \epsilon_k^2}$$

and closing the contour in the two possible ways.

Problem 2: Derive the $m^2T^2/24$ term of the high-T expansion directly from $f_T(m)$, and convince yourself that a naive Taylor expansion cannot produce the m^3T term.

The infrared problem

. . . but perturbation theory breaks down

The barrier was a *one-loop* effect. The expansion parameter is the one-loop term over the tree-level term:

$$\underbrace{\frac{T(3\lambda)^{3/2}|\bar{\phi}|^3}{12\pi}}_{\text{one loop, from } \omega_n = 0} \Big/ \underbrace{\frac{\lambda\bar{\phi}^4}{4}}_{\text{tree level}} \sim \frac{\lambda T}{\pi m_{\text{eff}}}\,, \qquad m_{\text{eff}} \sim \sqrt{\lambda}\,\bar{\phi}\,.$$

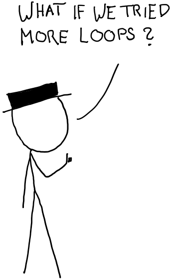
When it fails. For $m_{\text{eff}} \lesssim \lambda T/\pi$ the correction is as large as what it corrects. At the jump, $m_{\text{eff}} = \sqrt{3\lambda}\,\bar{\phi}_{\text{c}} = 3\lambda T/(2\pi)$: the barrier is exactly in this region.

Why. Bose enhancement, $n_{\text{B}} \simeq T/\epsilon_k$: each extra loop of soft modes costs $\lambda T/(\pi m_{\text{eff}})$, not λ.

Consequence. In $\mathrm{O}(N)$ scalar theory the transition is really of *second* order. The one-loop prediction is wrong.

Adding more loops does not help

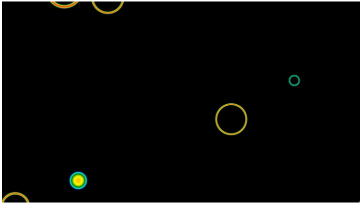
Resumming the "daisies" fixes the *leading* infrared divergence, but not the problem: at order g^6 the magnetic scale g^2T contributes at all loop orders at once.



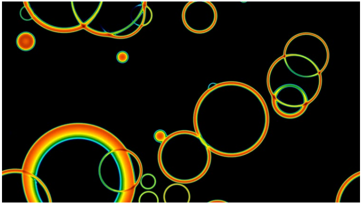
Two ways out: organize the expansion by scales (*dimensional reduction*), or treat the soft modes non-perturbatively (*lattice*). Lecture 2 combines both.

And this is what it looks like

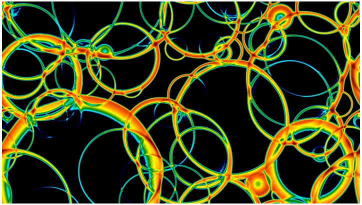
1. The barrier in $V_{\rm eff}$: thermal field theory. **2.** How many, and when: **lecture 2**. **3.** How fast, and what the fluid does: **lecture 3**.



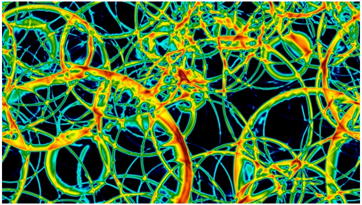
bubbles nucleate



and grow



they collide



and sound waves remain

[1] Simulation and video by D. Weir, `saoghal.net/slides/brda`; full movie `youtu.be/mfGL8Cp0RPA`

Summary of lecture 1

1: A phase transition is a non-analyticity of f. In field theory the order parameter is the expectation value of a scalar field, and the tool is the effective potential.

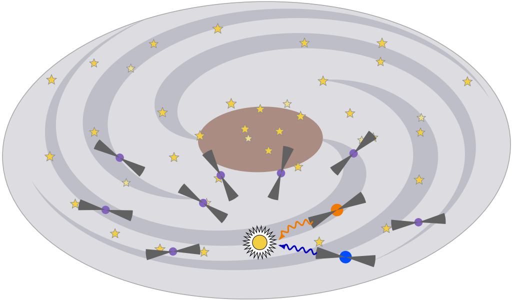
2: Thermal corrections restore the symmetry at high T, and at one loop the Matsubara zero mode adds a $-m^3T/12\pi$ term that makes the transition look *first order*.

3: That term is the tip of an infrared problem: at the ultrasoft scale g^2T perturbation theory fails at any finite loop order. The way out is a 3d effective theory *plus* the lattice.

Next: **that effective theory, the Standard Model, and models that do transition.**

Backup

And there may already be a hint

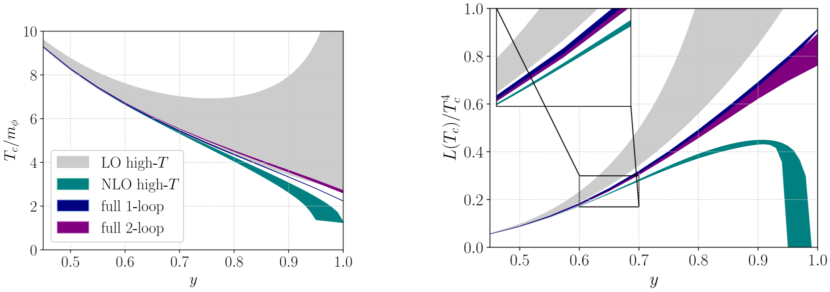


Pulsar timing arrays see a stochastic signal at nHz frequencies.[1] Supermassive black-hole binaries, or new physics? To answer that, the theory prediction must be under control.

[1] **NANOGrav** Collaboration, Afzal *et al.*, *The NANOGrav 15-year Data Set: Search for Signals from New Physics,* [2306.16219]

How accurately do we need the effective potential?

Critical temperature and latent heat at successive orders: the high-temperature expansion converges slowly, and the transition *strength* is the most sensitive quantity.[1]



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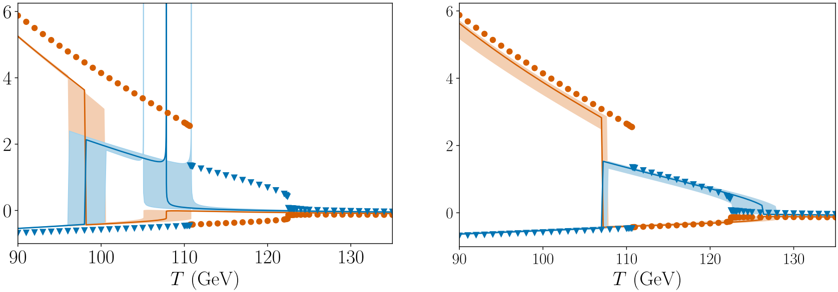
[1] Gould, Tenkanen, *On the perturbative expansion at high temperature and implications for cosmological phase transitions*, JHEP **06** (2021) 069 [2104.04399]

Convergence depends on the scale hierarchy assumed



Different power countings ("soft" vs. "supersoft" scalar mass) organize the same expansion differently. Where they disagree, the uncertainty comes from the EFT construction, not from the model.

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[11] Cominotti, Baroni, Rogora *et al.*, *Observation of Temperature Effects on False Vacuum Decay in Atomic Quantum Gases*, Phys. Rev. Lett. **135** (2025) 183401 [2504.03528].

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[13] Gould, Tenkanen, *On the perturbative expansion at high temperature and implications for cosmological phase transitions*, JHEP **06** (2021) 069 [2104.04399].