Gravitational waves from first-order phase transitions

Lecture 2: the 3d EFT, and nucleation

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Where we are

$$\begin{array}{c}\text{Lagrangian}\\ \{g, \lambda, y_t, \ldots\}\end{array} \longrightarrow \begin{array}{c}V_{\text{eff}}(\phi, T)\\ \text{3d EFT}\end{array} \longrightarrow T_{\rm n},\ \alpha,\ \beta/H_\star,\ v_w \longrightarrow \begin{array}{c}\Omega_{\rm GW}h^2(f)\\ \text{LISA}\end{array}$$

First half, equilibrium: the 3d EFT, the Standard Model, and a model with a tree-level barrier.

Second half, real time: the latent heat cannot be released everywhere at once, so the transition proceeds in three stages:

(i) nucleation of bubbles of the low-T phase (*today*);

(ii) growth of the nucleated bubbles (lecture 3);

(iii) collisions, sound waves, turbulence (lecture 3).

Recap of lecture 1, in one slide

The tool. The free energy of a scalar field is $V_{\text{eff}}(\phi, T)$, and thermal corrections restore the symmetry at high T.

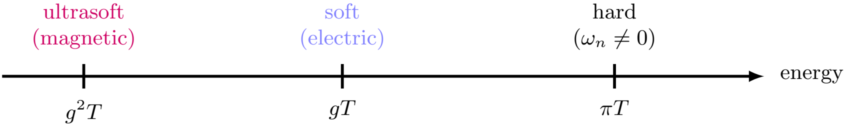
The apparent barrier. At one loop the *Matsubara zero mode* adds a $-m^3T/12\pi$ term, which makes the transition look *first order*.

The problem. That term is infrared sensitive: at the ultrasoft scale g^2T perturbation theory fails, and more loops make it worse.

> Today in two halves. First the equilibrium input: organize by scales, $\pi T \gg gT \gg g^2T$, reduce to a 3d EFT, see that the Standard Model only crosses over, and find a model with a *tree-level* barrier. Then **at which temperature, and at which rate, does the transition start?**

Dimensional reduction

A hierarchy of thermal scales



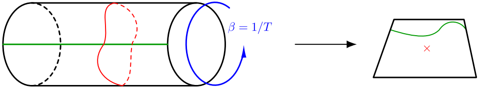
- **hard** πT: all non-zero Matsubara modes $\Rightarrow$ integrate out, perturbatively;
- **soft** gT: the temporal gauge fields A_0^a, B_0, screened by Debye masses $\Rightarrow$ integrate out, perturbatively;
- **ultrasoft** g^2T: spatial gauge fields and the Higgs $\Rightarrow$ non-perturbative, put on the lattice.

Turning the tables: the zero modes are the problem
$\Rightarrow$ make them the degrees of freedom

- For small λ, $(\lambda T/\pi)^2 \ll (2\pi T)^2$: all modes with $\omega_n \neq 0$ carry a hard mass $\geq 2\pi T$, so only $\omega_n = 0$ is infrared sensitive.
- Zero modes do not depend on τ: they live in $d = 3$. This is *dimensional reduction.*

$$S_{\text{eff}} = \frac{1}{T}\int \mathrm{d}^3x \Big\{ \tfrac{1}{2}(\partial_i\phi_3)^2 + \tfrac{1}{2}m_3^2\phi_3^2 + \tfrac{1}{4}\lambda_3\phi_3^4 + \ldots \Big\}\,.$$

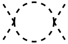
The 3d parameters m_3^2, λ_3 are fixed by *matching* to the full theory (next slide).



Matching: the 3d parameters from hard loops

Demand that the *static* correlators of the 3d EFT reproduce those of the full theory at distances $\gg 1/(\pi T)$. The 4d loops contain only the hard modes $\omega_n \neq 0$; the zero mode *is* the 3d field.[1]

2-point: $m_3^2 = -m^2 +$  $\Big|_{\omega_n \neq 0} = -m^2 + \dfrac{\lambda T^2}{4} + \ldots$

4-point: $\lambda_3 = T\Big[\lambda +$  $\Big|_{\omega_n \neq 0} + \ldots\Big]$

The tadpole is worked out in the backup.

Two loops, any model: automated in `DRalgo`.[2]

[1] Kajantie, Laine, Rummukainen, Shaposhnikov, *Generic rules for high temperature dimensional reduction and their application to the standard model,* Nucl. Phys. B **458** (1996) 90 [hep-ph/9508379]

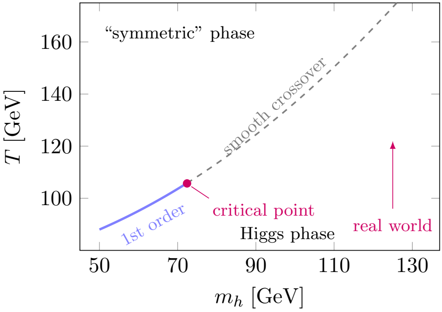
[2] Ekstedt, Schicho, Tenkanen, `DRalgo`*: A package for effective field theory approach for thermal phase transitions,* Comput. Phys. Commun. **288** (2023) 108725 [2205.08815]

The Standard Model

The Standard Model: a crossover, not a phase transition

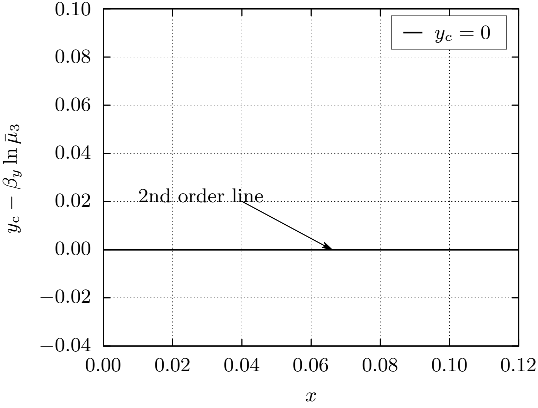
$$\mathcal{L}_{\text{eff}} = \tfrac{1}{4}G^a_{ij}G^a_{ij} + \tfrac{1}{4}F_{ij}F_{ij} + (D_i\phi_3)^\dagger(D_i\phi_3) + m_3^2\phi_3^\dagger\phi_3 + \lambda_3(\phi_3^\dagger\phi_3)^2 + \dots$$

UV side: parameters known to two loops from matching.
IR side: solved non-perturbatively on the lattice.
Result: the first-order line ends at $m_h \simeq 72$ GeV, so for $m_h = 125$ GeV the Standard Model has a smooth crossover.[1]



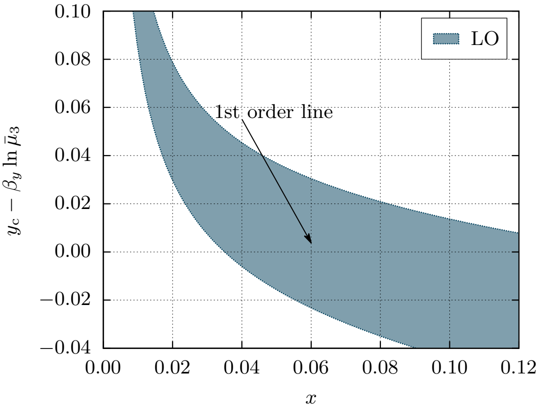
[1] Kajantie, Laine, Rummukainen, Shaposhnikov, *Is there a hot electroweak phase transition at $m_H \gtrsim m_W$?*, Phys. Rev. Lett. **77** (1996) 2887 [hep-ph/9605288]

Lattice vs. perturbation theory in the 3d EFT



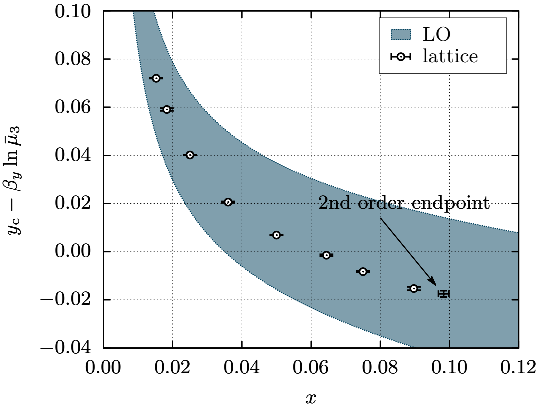
The 3d SU(2)+Higgs theory has a two-dimensional parameter space.

Lattice vs. perturbation theory in the 3d EFT



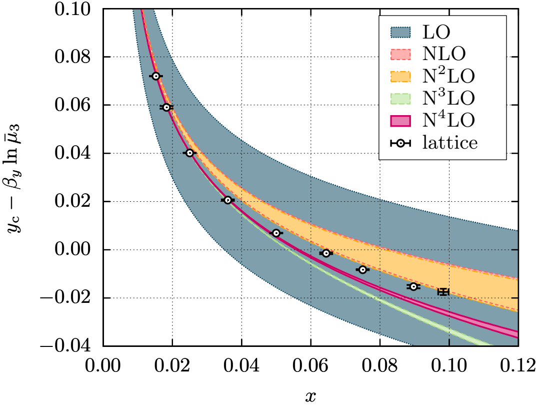
Perturbation theory at leading order predicts a transition line everywhere.

Lattice vs. perturbation theory in the 3d EFT



The lattice finds that the line ends.

Lattice vs. perturbation theory in the 3d EFT



Beyond the endpoint: a crossover. This is the Standard Model.

Problem

Problem 1: Estimate T_c for the Standard Model from the one-loop high-T potential, and compare with the lattice value $\simeq 160$ GeV.

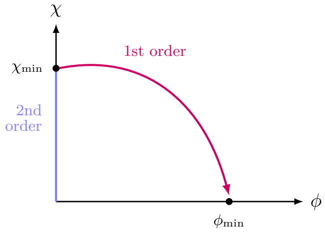
Beyond the Standard Model

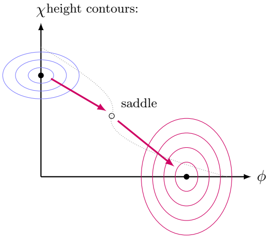
Beyond the Standard Model: a strong transition

$$V = -\frac{m^2\phi^2}{2} + \frac{\lambda\phi^4}{4} - \frac{M^2\chi^2}{2} + \frac{\kappa\chi^4}{4} + \frac{\gamma\phi^2\chi^2}{2}\,,$$

thermal masses $m_3^2 = -m^2 + \frac{(3\lambda+\gamma)T^2}{12}$, $M_3^2 = -M^2 + \frac{(3\kappa+\gamma)T^2}{12}$.

If $T_\chi > T_\phi$: a two-step transition (left). First χ condenses (second order), then the system jumps to $(\phi_{\min}, 0)$ (first order) through a *tree-level* saddle (right): parametrically stronger, and free of the IR problem.





How large do these numbers have to be?

Strength

$$\alpha \simeq \frac{1}{\rho_{\rm rad}}\left[\Delta V_{\rm eff} - \tfrac{T}{4}\partial_T \Delta V_{\rm eff}\right]_{T_{\rm n}}$$

α: the fraction of the radiation energy $\rho_{\rm rad}$ that the transition releases; $\Delta V_{\rm eff}$ is the free-energy gap between the phases.

Inverse duration

$$\frac{\beta}{H_\star} = T\frac{{\rm d}S_{\rm E}}{{\rm d}T}\bigg|_{T_{\rm n}}$$

$\beta/H_\star$: how fast the transition runs compared with the Hubble rate $H_\star$; $S_{\rm E}$ is the bounce action (**later today**).

With the bubble spacing $\ell_{\rm B} \simeq (8\pi)^{1/3} v_w/\beta$ (v_w from **lecture 3**), only one combination reaches the signal:

$$\Omega_{\rm GW}h^2 \lesssim 10^{-5}\,\alpha^2\,(\ell_{\rm B}H_\star)^2 \;\propto\; \left[\frac{\alpha\,H_\star}{\beta}\right]^2.$$

Loud needs *strong* (large α) and *slow* (small $\beta/H_\star$).

How large do these numbers have to be?

Strength

$$\alpha \simeq \frac{1}{\rho_{\mathrm{rad}}}\left[\Delta V_{\mathrm{eff}} - \tfrac{T}{4}\partial_T \Delta V_{\mathrm{eff}}\right]_{T_{\mathrm{n}}}$$

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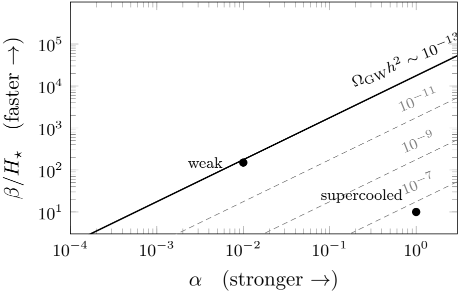

Lines of constant $\Omega_{\mathrm{GW}}h^2$ at $v_w = 0.6$ (an order of magnitude, not a sensitivity curve). A weak electroweak transition reaches 10^{-13}, a supercooled one six orders more.

How large do these numbers have to be?

Strength

$$\alpha \simeq \frac{1}{\rho_{\mathrm{rad}}}\left[\Delta V_{\mathrm{eff}} - \frac{T}{4}\partial_T \Delta V_{\mathrm{eff}}\right]_{T_{\mathrm{n}}}$$

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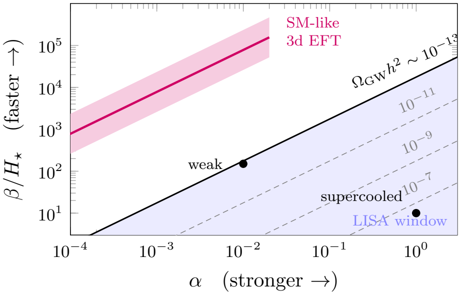


An *SM-like* 3d EFT cannot leave the wrong corner: a loud signal needs *light* new states.[1]

[1] Gould, Kozaczuk, Niemi, Ramsey-Musolf, Tenkanen, Weir, *Nonperturbative analysis of the gravitational waves from a first-order electroweak phase transition*, Phys. Rev. D **100** (2019) 115024 [1903.11604]

Problem 2: In the two-scalar model, show that the conditions $T_\chi > T_\phi$ and $V_{\rm eff}(\phi_{\rm min}, 0) < V_{\rm eff}(0, \chi_{\rm min})$, together with vacuum stability, carve out a non-empty region of the $(\lambda/\gamma, \kappa/\gamma)$ plane. Sketch it.

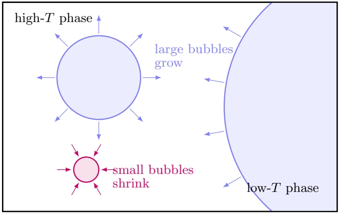
What is nucleation?

Supercooling and bubbles

At $T < T_c$ the high-T phase is *metastable*: its free energy is higher by $\Delta f > 0$. Thermal fluctuations create bubbles of the low-T phase:

- small bubbles shrink: the surface tension wins;
- large bubbles grow: the volume energy wins.

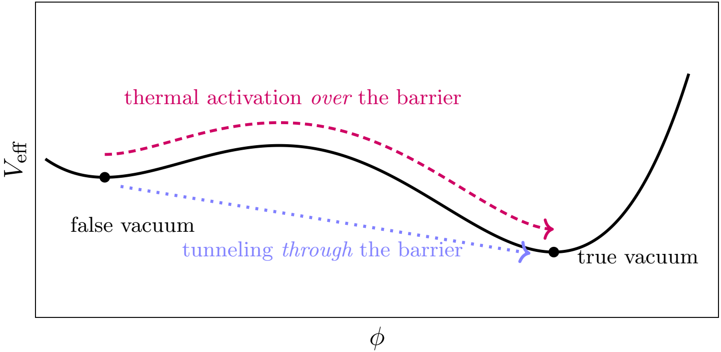
The **nucleation rate** Γ is the number of critical bubbles formed per unit volume and time.



$T < T_c$ (supercooling)

Two pictures, one description

Thermal activation over the barrier, or quantum tunneling through it? Two limits of one path-integral description.



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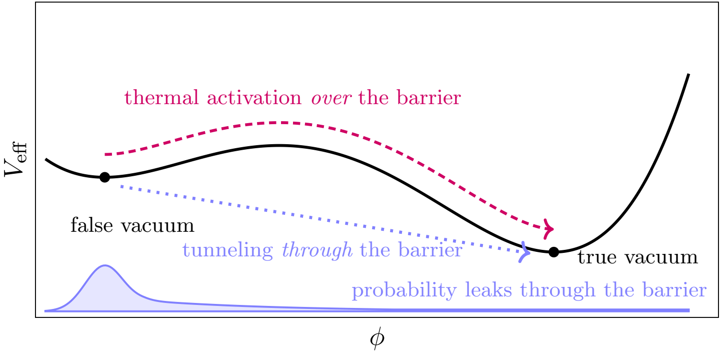
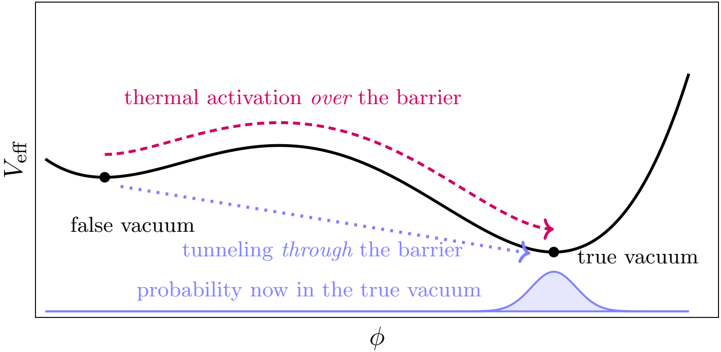


Tunneling: the distribution localized in the metastable vacuum has a tail that leaks through the barrier.

Two pictures, one description

Thermal activation over the barrier, or quantum tunneling through it? Two limits of one path-integral description.



Eventually the probability sits in the true vacuum. The probability flowing out per unit time is the decay rate.

The bounce

Both limits are described by a *bounce* $\hat{\phi}(\tau, \mathbf{x})$:[1]

- a solution of the Euclidean equation of motion,

$$\left.\frac{\delta S_{\mathrm{E}}}{\delta \phi}\right|_{\hat{\phi}} = 0\,,$$

  periodic in τ and decaying at large $|\mathbf{x}|$;
- a *saddle point* of S_{E}: the fluctuation operator $\delta^2 S_{\mathrm{E}}/\delta\phi^2$ has one negative eigenvalue and three zero eigenvalues.

Zero modes $\to$ the volume factor in Γ
negative mode $\to$ the decay.

[1] Coleman, *Fate of the false vacuum: Semiclassical theory,* Phys. Rev. D **15** (1977) 2929

Zero modes are the volume factor

Differentiating the equation of motion with respect to x_i,

$$\frac{\partial}{\partial x_i}\left\{ \left.\frac{\delta S_{\mathrm{E}}}{\delta \phi}\right|_{\hat{\phi}} = 0 \right\} \quad \Longrightarrow \quad \left.\frac{\delta^2 S_{\mathrm{E}}}{\delta \phi^2}\right|_{\hat{\phi}} \partial_i \hat{\phi} = 0\,.$$

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The three zero modes correspond to *translations*: the bubble can nucleate anywhere, with the same action. Treating them as collective coordinates produces

$$\Gamma \;\propto\; V \quad\Longrightarrow\quad \Gamma/V \equiv \text{rate per unit volume}\,.$$

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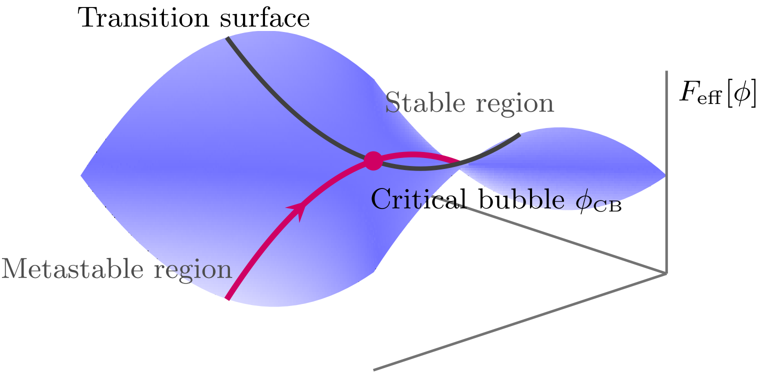
The single *negative* mode is what makes the configuration a saddle, and what gives the rate an imaginary part, hence a decay.

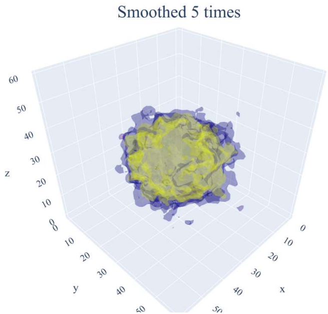
The bounce in configuration space



the bounce is the saddle point between the two phases

The nucleation rate is $\Gamma = A\, e^{-S_{\text{E}}}$, with S_{E} the bounce action and A a fluctuation determinant (statistical $\times$ dynamical prefactor).

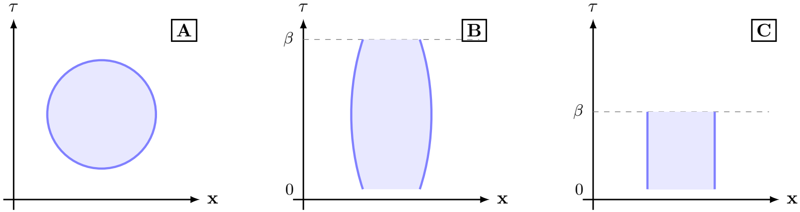
The bounce in configuration space



at the saddle: the critical bubble

The nucleation rate is $\Gamma = A\, e^{-S_{\mathrm{E}}}$, with S_{E} the bounce action and A a fluctuation determinant (statistical $\times$ dynamical prefactor).

Symmetry of the bounce: instanton, caloron, sphaleron



$T = 0$

$\Rightarrow$ 4d symmetry

$\Rightarrow$ "instanton" for quantum tunneling

$T \neq 0$

$\Rightarrow$ (3+1)d symmetry

$\Rightarrow$ "caloron" for thermally modified quantum tunneling

$T \gg 0$

$\Rightarrow$ 3d symmetry

$\Rightarrow$ classical thermal tunneling, "sphaleron"

A is the $O(4)$-symmetric instanton of vacuum decay.

B is the caloron that interpolates as the compact direction shrinks.

C is the relevant bounce in cosmology: it is τ-independent, and $S_{\mathrm{E}}[\hat{\phi}] = \beta \int \mathrm{d}^3x \, \mathcal{L}_{\mathrm{E}}$.

The classical estimate

Critical radius from classical thermodynamics

Expanding $\Delta f(T) = L(1 - T/T_\mathrm{c}) + \mathcal{O}((T - T_\mathrm{c})^2)$ around T_c, with L the latent heat, a bubble of radius R costs

$$S_\mathrm{E} = \frac{\Delta F}{T} = \frac{1}{T}\Big\{\underbrace{-L\Big(1 - \frac{T}{T_\mathrm{c}}\Big)\frac{4\pi R^3}{3}}_{\text{gain}} + \underbrace{\sigma\, 4\pi R^2}_{\text{cost}}\Big\},$$

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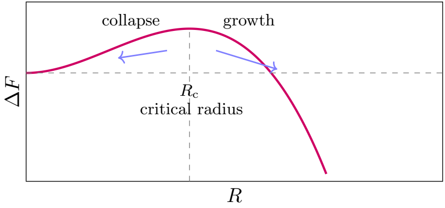
which is maximized at

$$R_\text{c} = \frac{2\sigma}{L(1 - \frac{T}{T_\text{c}})}\,,$$

$$\Delta F = \frac{16\pi\sigma^3}{3L^2(1 - \frac{T}{T_\text{c}})^2}\,.$$



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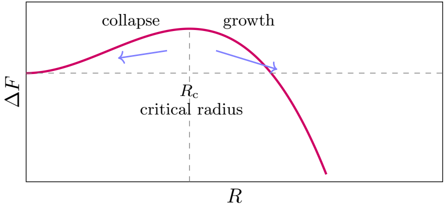


$T \to T_\mathrm{c}$: $S_\mathrm{E} \to \infty$, $\Gamma \to 0$, no bubbles $\Rightarrow$ supercooling.

The surface tension in field theory

The bounce solves the spherical equation of motion

$$\frac{\mathrm{d}^2\hat{\phi}}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}r} = V'_{\mathrm{eff}}(\hat{\phi})\,, \qquad \hat{\phi}'(0) = 0\,, \quad \hat{\phi}(\infty) = 0\,.$$

Read r as a time: $\hat{\phi}$ is a particle rolling in $-V_{\mathrm{eff}}$ with friction $2/r$, released at rest near ϕ_{b} and arriving at rest at the origin.

Thin wall ($T \to T_{\mathrm{c}}$): $R_{\mathrm{c}} \to \infty$, so across the wall $2/r \simeq 2/R_{\mathrm{c}} \to 0$ and the friction term drops out.

The surface tension in field theory

The bounce solves the spherical equation of motion

$$\frac{\mathrm{d}^2\hat{\phi}}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}r} = V_{\mathrm{eff}}'(\hat{\phi})\,, \qquad \hat{\phi}'(0) = 0\,, \quad \hat{\phi}(\infty) = 0\,.$$

1. First integral. Without friction, multiply by $\hat{\phi}'$ and integrate ("energy conservation"), with $\hat{\phi}' = 0$ outside the wall:

$$\tfrac{1}{2}\hat{\phi}'^2 = V_{\mathrm{eff}}(\hat{\phi}) - V_{\mathrm{eff}}(\phi_{\mathrm{b}}) \equiv \Delta V(\hat{\phi})\,.$$

2. Surface tension = energy per wall area; kinetic and potential terms are equal, and $\mathrm{d}r = \mathrm{d}\phi/\hat{\phi}'$:

$$\sigma = \int \mathrm{d}r \left[\tfrac{1}{2}\hat{\phi}'^2 + \Delta V\right] = \int \mathrm{d}r\, \hat{\phi}'^2 = \int_0^{\phi_{\mathrm{b}}} \mathrm{d}\phi\, \sqrt{2\,\Delta V(\phi)}\,.$$

The surface tension in field theory

The bounce solves the spherical equation of motion

$$\frac{\mathrm{d}^2\hat{\phi}}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}r} = V'_{\mathrm{eff}}(\hat{\phi})\,, \qquad \hat{\phi}'(0) = 0\,, \quad \hat{\phi}(\infty) = 0\,.$$

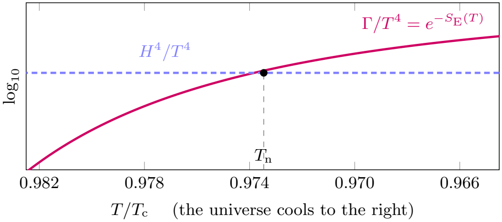
Away from T_{c} the thin wall fails and the bounce is solved numerically.

For a *tree-level* barrier the wall is thick, and the thin-wall S_{E} can be off by an order of magnitude.

Defining the nucleation temperature

One bubble per Hubble volume per Hubble time. The rate rises steeply as the universe supercools, until it beats the expansion:

$$\frac{\Gamma(T_\mathrm{n})}{V} \simeq H^4(T_\mathrm{n}) \qquad \Leftrightarrow \qquad S_\mathrm{E}(T_\mathrm{n}) \simeq 4\ln\frac{T_\mathrm{n}}{H(T_\mathrm{n})} \simeq 140\,.$$



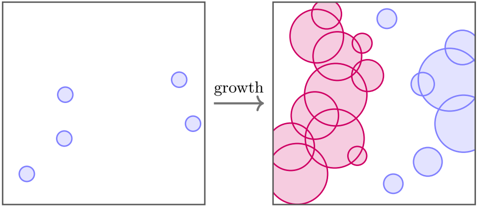
A logarithm, hence robust: it moves by 4 per e-fold of T_n. This is the *onset*, with the universe still in the old phase.[1]

[1] Laine, Vuorinen, *Basics of Thermal Field Theory*, vol. 925 of *Lecture Notes in Physics*. Springer International Publishing, Cham, Jan, 2016, [1701.01554]

Defining the nucleation temperature

More carefully one asks for *percolation*, with h the fraction of space still in the old phase:

$$h(T) = \exp\left\{ -\frac{4\pi}{3} \int_T^{T_c} \frac{dT'\,\Gamma(T')}{T'^4 H(T')} \left(\int_T^{T'} \frac{dT''\,v_w}{H(T'')} \right)^3 \right\}.$$



$T \simeq T_n$:
isolated critical bubbles, $h \simeq 1$

$T = T_p$:
one cluster spans the box, $h \simeq 0.71$

Percolation is when the connected cluster first spans the volume.[1]

[1] Hindmarsh, Lüben, Lumma, Pauly, *Phase transitions in the early universe*, SciPost Phys. Lect. Notes **24** (2021) 1 [2008.09136]. v_w enters only through the swept volume $\propto v_w^3$, hence only inside a logarithm, and shifts T_n by a per cent. For the *signal* it is not small: $\ell_B = (8\pi)^{1/3} v_w/\beta$ sets the peak frequency (lecture 3).

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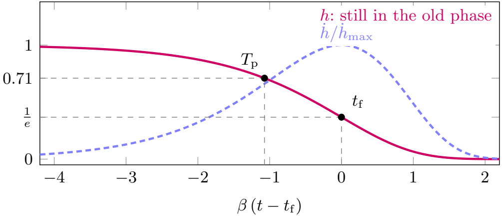
More carefully one asks for *percolation*, with h the fraction of space still in the old phase:

$$h(T) = \exp\left\{ -\frac{4\pi}{3} \int_T^{T_\mathrm{c}} \frac{\mathrm{d}T'\,\Gamma(T')}{T'^4 H(T')} \left(\int_T^{T'} \frac{\mathrm{d}T''\, v_w}{H(T'')} \right)^3 \right\}.$$



h falls to $1/e$ at t_f; the connected cluster spans space at $h \simeq 0.71$, which defines T_p.[1]

[1] Hindmarsh, Lüben, Lumma, Pauly, *Phase transitions in the early universe*, SciPost Phys. Lect. Notes **24** (2021) 1 [2008.09136]. v_w enters only through the swept volume $\propto v_w^3$, hence only inside a logarithm, and shifts T_n by a per cent. For the *signal* it is not small: $\ell_\mathrm{B} = (8\pi)^{1/3} v_w/\beta$ sets the peak frequency (lecture 3).

Problems

Problem 3: Starting from the spherical equation of motion, derive the thin-wall expression for the surface tension,

$$\sigma = \int_0^{\phi_{\mathrm{b}}} \mathrm{d}\phi \, \sqrt{2\big[V_{\mathrm{eff}}(\phi) - V_{\mathrm{eff}}(\phi_{\mathrm{b}})\big]} \, .$$

Problem 4: Show that extremising

$$S_{\mathrm{E}}(R) = \frac{1}{T}\left[-\Delta f \, \frac{4\pi R^3}{3} + \sigma \, 4\pi R^2\right]$$

gives $R_{\mathrm{c}} = 2\sigma/\Delta f$ and $S_{\mathrm{E}}(R_{\mathrm{c}}) = 16\pi\sigma^3/(3\Delta f^2 T)$.

Computing the rate

The rate beyond the leading exponential

At $T = 0$: $\Gamma = -2\,\mathrm{Im}\,E$. At finite T, $\Gamma \overset{?}{\simeq} -2\,\mathrm{Im}\,F$ misses a real-time ingredient: a Euclidean free energy knows nothing about how fast the bubble grows. Langer's factor $|\lambda_-|/\pi$ supplies it:

$$\Gamma = \underbrace{\frac{|\lambda_-|}{\pi}}_{\text{dynamical}} \; \underbrace{\frac{\mathrm{Im}\,\mathcal{Z}_{\text{bounce}}}{\mathcal{Z}_{\text{sym}}}}_{\text{statistical}} \sim A\, e^{-S_{\mathrm{E}}}\,.$$

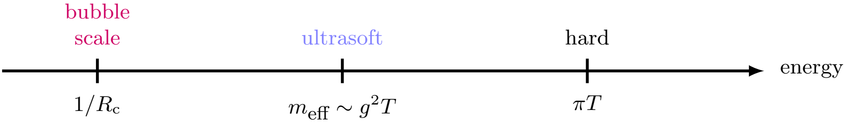
Three problems:

- the bounce is a saddle of the *effective* action: IR problem;
- zero and negative modes must be removed from the determinant;
- the result must not depend on the scale $\bar{\mu}$.

Modern approach: compute inside the 3d EFT.[1]

[1] Gould, Hirvonen, *Effective field theory approach to thermal bubble nucleation*, Phys. Rev. D **104** (2021) 096015 [2108.04377]

Nucleation has its own scale hierarchy



New scale. The critical bubble is *larger* than the correlation length, $R_c \gg 1/m_{\text{eff}}$: one more step below the ultrasoft scale.

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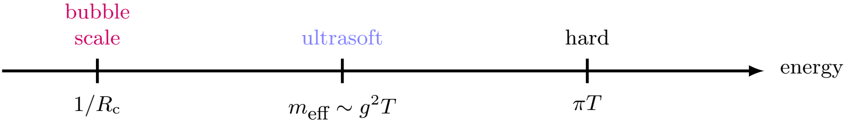

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Consequence. Fluctuations around the bounce live in the 3d EFT. The zero modes of the *bounce* (not of the vacuum) are treated separately.

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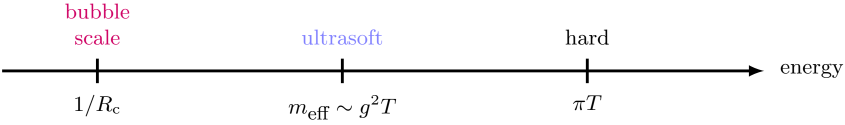

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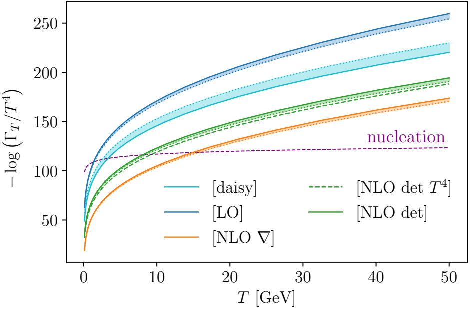
Consequence. Fluctuations around the bounce live in the 3d EFT. The zero modes of the *bounce* (not of the vacuum) are treated separately.

Consistency. Count powers of g the same way in V_{eff} and in the determinant, or the $\bar{\mu}$ dependence does not cancel (next slide).

Scale dependence is a diagnostic



Plotted: $-\ln(\Gamma/T^4)$ against T; each band varies $\bar{\mu}$ by a factor two around πT.[1]

Read: the width of a band estimates the missing higher orders.

[1] Croon, Gould, Schicho, Tenkanen, White, *Theoretical uncertainties for cosmological first-order phase transitions,* JHEP **04** (2021) 055 [2009.10080]; Hirvonen, Löfgren, Ramsey-Musolf, Schicho, Tenkanen, *Computing the gauge-invariant bubble nucleation rate in finite temperature effective field theory,* JHEP **07** (2022) 135 [2112.08912]

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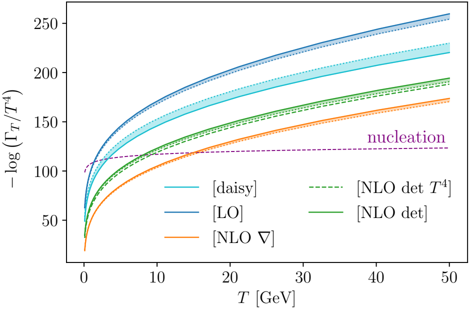

Plotted: $-\ln(\Gamma/T^4)$ against T; each band varies $\bar{\mu}$ by a factor two around πT.[1]

Expect: bands *shrink* order by order, since $\bar{\mu}\,\partial_{\bar{\mu}} \ln \Gamma = \mathcal{O}(\sqrt{x})$.

---

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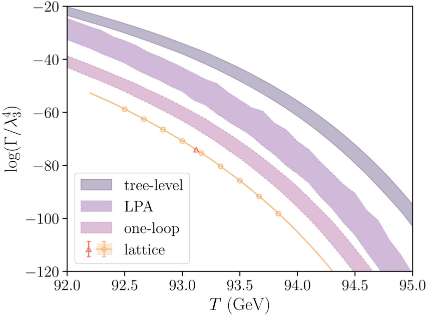
Successive orders in the nucleation rate

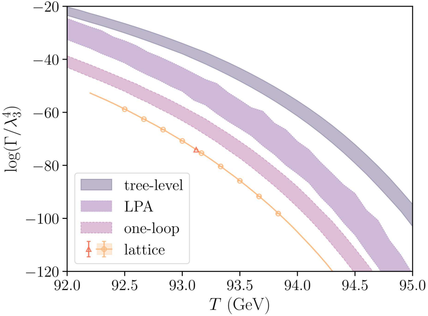


Plotted: $\log(\Gamma/\lambda_3^4)$ at tree level, in the local-potential approximation and at one loop, against the lattice.[1]

Read: each order shifts $\log\Gamma$ by $\mathcal{O}(10)$, i.e. the rate by orders of magnitude.

[1] Gould, Kormu, Weir, *Nonperturbative test of nucleation calculations for strong phase transitions,* Phys. Rev. D **111** (2025) L051901 [2404.01876]

Successive orders in the nucleation rate



Plotted: $\log(\Gamma/\lambda_3^4)$ at tree level, in the local-potential approximation and at one loop, against the lattice.[1]

But: T_n follows from $S_\text{E}(T_\text{n}) \simeq 140$, so it moves only logarithmically.

[1] Gould, Kormu, Weir, *Nonperturbative test of nucleation calculations for strong phase transitions,* Phys. Rev. D **111** (2025) L051901 [2404.01876]

Determining the prefactor: three equations of motion

1. The bounce, a saddle of the nucleation-scale action, solved by shooting or gradient flow:

$$-\nabla^2\hat{\phi} + S'_{\text{nucl}}(\hat{\phi}) = 0\,, \qquad \hat{\phi}'(0) = 0\,, \quad \hat{\phi}(\infty) = 0\,.$$

2. The fluctuation operator, one eigenvalue problem per partial wave, solved by Gel'fand–Yaglom (`BubbleDet`):

$$\left[-\nabla^2 + S''_{\text{nucl}}(\hat{\phi})\right] f_n = \lambda_n f_n\,, \qquad \text{one } \lambda_- < 0\,, \quad \text{three } \lambda = 0\,.$$

3. The real-time Langevin equation for the growth rate κ of the critical bubble (next section), with η the friction:

$$\kappa = \sqrt{|\lambda_-| + \eta^2/4} - \frac{\eta}{2} \quad \xrightarrow[\eta\to 0]{} \quad \sqrt{|\lambda_-|}\,, \qquad \xrightarrow[\eta\gg\sqrt{|\lambda_-|}]{} \quad \frac{|\lambda_-|}{\eta}\,.$$

Steps 1 and 2 give the equilibrium rate; step 3 is usually left uncomputed.[1]

[1] Ekstedt, Gould, Hirvonen, *BubbleDet: a Python package to compute functional determinants for bubble nucleation*, JHEP **12** (2023) 056 [2308.15652]; Gould, Hirvonen, *Effective field theory approach to thermal bubble nucleation*, Phys. Rev. D **104** (2021) 096015 [2108.04377]

Energy against entropy: where the logarithms come from

Like any thermal rate: (number of ways) $\times$ (Boltzmann factor),[1]

$$\Gamma \sim \underbrace{e^{S}}_{\#\text{ critical bubbles}} \times \underbrace{e^{-E_\mathrm{c}/T}}_{\text{cost of one}} \quad \Longrightarrow \quad -\ln\Gamma = \frac{E_\mathrm{c}}{T} - S\,,$$

with $E_\mathrm{c} = \hat{S}_{3\mathrm{d}}$ the energy of the critical bubble.

[1] Laine, Vuorinen, *Basics of Thermal Field Theory*, vol. 925 of *Lecture Notes in Physics*. Springer International Publishing, Cham, Jan, 2016, [1701.01554]; Hirvonen, Löfgren, Ramsey-Musolf, Schicho, Tenkanen, *Computing the gauge-invariant bubble nucleation rate in finite temperature effective field theory,* JHEP **07** (2022) 135 [2112.08912]

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- **where** it sits: anywhere in the volume, $S \supset \ln V$;
- **how** it wiggles: every fluctuation mode, one ODE per partial wave, $S \supset -\frac{1}{2}\ln\big(\det'[-\nabla^2 + V''(\hat{\phi})]/\det[-\nabla^2 + V''(0)]\big)$;

---

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- stiff modes (large eigenvalues): few configurations, less entropy.

[1] Laine, Vuorinen, *Basics of Thermal Field Theory*, vol. 925 of *Lecture Notes in Physics*. Springer International Publishing, Cham, Jan, 2016, [1701.01554]; Hirvonen, Löfgren, Ramsey-Musolf, Schicho, Tenkanen, *Computing the gauge-invariant bubble nucleation rate in finite temperature effective field theory*, JHEP **07** (2022) 135 [2112.08912]

Problems

Problem 5: Using $\Gamma \simeq T^4 e^{-S_{\rm E}}$ and $H^2 = \rho/(3M_{\rm Pl}^2)$, verify that $\Gamma \simeq H^4$ corresponds to $S_{\rm E} \simeq 4\ln(M_{\rm Pl}/T) \simeq 140$ at $T \sim 100$ GeV.

Problem 6: Show that $\beta/H_\star = T\,\mathrm{d}S_{\rm E}/\mathrm{d}T$ follows from linearizing $S_{\rm E}(t)$ around $T_{\rm n}$, and estimate it for the classical $S_{\rm E}(T)$ above.

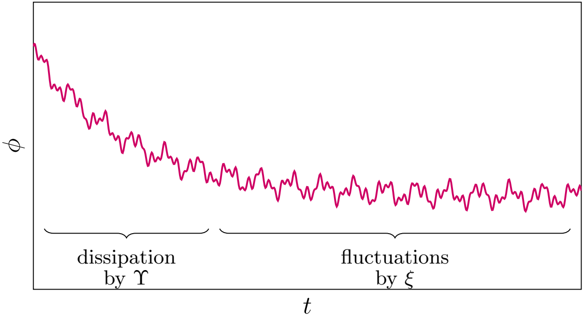
Real-time picture

Nucleation as a Langevin process

Fluctuating hydrodynamics for T, u^μ and the order parameter ϕ,

$$\partial^\mu\partial_\mu\phi + \underbrace{\Upsilon\, u^\mu\partial_\mu\phi}_{\text{friction}} + V'_{\text{eff}}(\phi) = \underbrace{\xi}_{\text{noise}}\ , \quad \langle\xi(X)\xi(Y)\rangle = \Omega\,\delta^{(4)}(X-Y)\,.$$

Computing the equal-time correlator and matching onto the $\omega_n = 0$ propagator T/ϵ_k^2 fixes $\Omega = 2\Upsilon T$ (fluctuation–dissipation).



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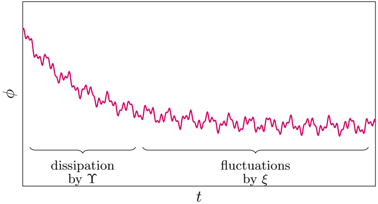
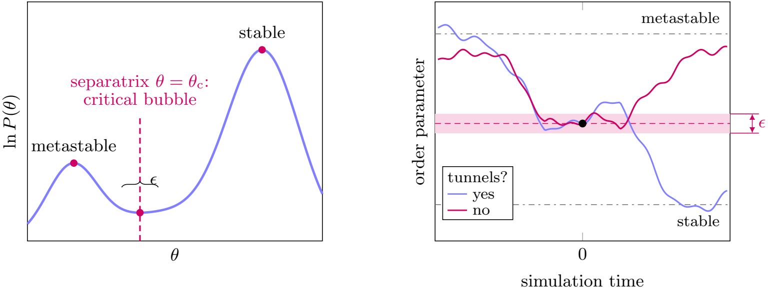


Noise drives nucleation. Friction is a local model of the out-of-equilibrium pressure, one of the forces that fix v_w (lecture 3). Both come from the same Υ.

What the lattice actually measures

Langer: the rate is the *probability* of being on the separatrix $\theta = \theta_c$, times the mean *flux* across it, corrected for recrossings:[1]

$$\Gamma\mathcal{V} \simeq \frac{1}{2} \underbrace{P_c}_{\text{multicanonical}} \underbrace{\langle \text{flux} \rangle}_{\text{equilibrium}} \underbrace{\langle \mathbf{d} \rangle}_{\text{real time}} .$$



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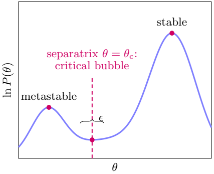
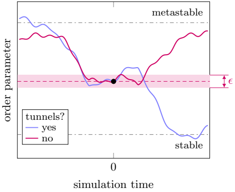


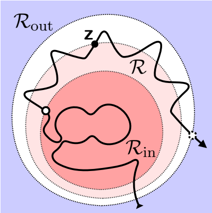


θ: order parameter, from the volume-averaged field.
$\mathbf{d} \leq 1$: fraction of crossings that end in the stable phase.

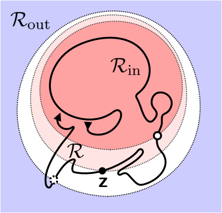
[1] Gould, Kormu, Weir, *Nonperturbative test of nucleation calculations for strong phase transitions*, Phys. Rev. D **111** (2025) L051901 [2404.01876]

Not every crossing is a decay

Follow single trajectories through phase space.[1]



(a) sloshing in the false vacuum



(b) sloshing in the true vacuum

Sloshing: a trajectory crosses $\partial\mathcal{R}$ at $\mathbf{z}$, makes a large excursion, and crosses again. Counted in $\mathbf{d}$, but it is not a decay at $\mathbf{z}$.

[1] Figure from Gould, Hirvonen, Shkerin, Sibiryakov, *Dynamics of nucleation in thermal phase transitions,* [2608.00169]

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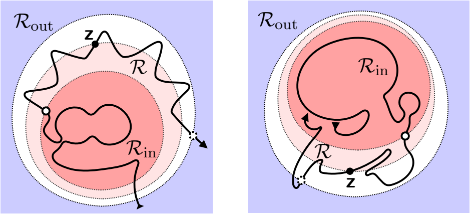


(a) sloshing in the false vacuum (b) sloshing in the true vacuum

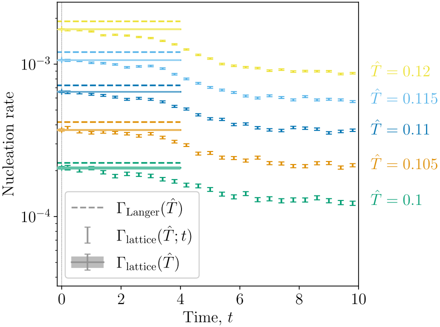
Sloshing: a trajectory crosses $\partial\mathcal{R}$ at $\mathbf{z}$, makes a large excursion, and crosses again. Counted in $\mathbf{d}$, but it is not a decay at $\mathbf{z}$.

In field theory: (a) happens, (b) does not, since a supercritical bubble keeps growing. Removing (a) gives the steady-state rate.

[1] Figure from Gould, Hirvonen, Shkerin, Sibiryakov, *Dynamics of nucleation in thermal phase transitions,* [2608.00169]

Measuring the rate without perturbation theory

Lattice: $P_{\rm c}$ multicanonically, $\mathbf{d}$ from real-time evolution.[1]



Plotted: rate against time since preparation; dashed: Langer.[2]

[1] Moore, Rummukainen, *Electroweak bubble nucleation, nonperturbatively,* Phys. Rev. D **63** (2001) 045002 [0009132]

[2] Hirvonen, Gould, *Langer's Nucleation Rate Reproduced on the Lattice,* Phys. Rev. Lett. **136** (2026) 081601 [2505.22732]; Hirvonen, *Real-time nucleation and off-equilibrium effects in high-temperature quantum field theories,* Phys. Rev. D **111** (2025) 116020 [2403.07987]

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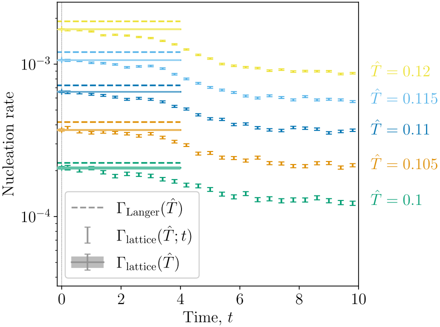

Read: rate depends on t; Langer holds to $\simeq 10\%$ at $t \to 0$.

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[2] Hirvonen, Gould, *Langer's Nucleation Rate Reproduced on the Lattice*, Phys. Rev. Lett. **136** (2026) 081601 [2505.22732]; Hirvonen, *Real-time nucleation and off-equilibrium effects in high-temperature quantum field theories,* Phys. Rev. D **111** (2025) 116020 [2403.07987]

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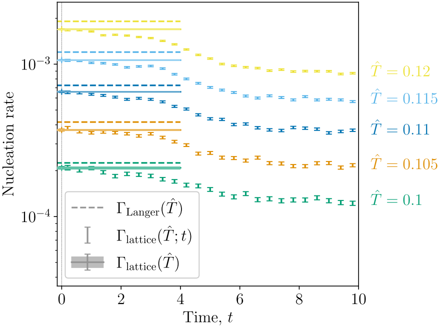

So: Langer is right *given* equilibrium; a real transition need not be.

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Summary of lecture 2

1: Nucleation is a saddle point of the effective action: the bounce. At high T it is three-dimensional; its zero modes give the volume factor, its negative mode the decay.

2: Classical thermodynamics gives $S_{\mathrm{E}} = 16\pi\sigma^3/[3L^2(1 - T/T_{\mathrm{c}})^2 T]$: the action diverges at T_{c}, so nucleation requires supercooling, and T_{n} is defined by $S_{\mathrm{E}}(T_{\mathrm{n}}) \simeq 140$.

3: Because S_{E} appears in an exponent, T_{n} is the largest single source of theory uncertainty in the predicted gravitational-wave signal.

Equilibrium: T_{c}, L, σ. *Nucleation*: T_{n}, ℓ_{B}, v_w.

Next: **how fast does the bubble wall go, and what does it radiate?**

Backup

Matching in practice: one example

The one-loop contribution to m_3^2 from $\frac{1}{4}\lambda\phi^4$ (combinatorial factor $4 \cdot 3/4 = 3$):

$$\delta m_3^2 = 3\lambda \sum\!\!\!\!\!\!\!\!\int_K' \frac{1}{\omega_n^2 + k^2} \,,$$

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where the prime removes $\omega_n = 0$ (it belongs to the EFT). Using the sum rule and dropping scaleless pieces in dimensional regularisation,

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k} \frac{1}{e^{k/T} - 1} = \frac{2T^{d-1}\Gamma(d-1)\zeta(d-1)}{(4\pi)^{d/2}\Gamma(\frac{d}{2})} \overset{d=3}{=} \frac{T^2}{12}$$
$$\implies \quad \delta m_3^2 = \frac{\lambda T^2}{4}\,.$$

[1] Ekstedt, Schicho, Tenkanen, DRalgo*: A package for effective field theory approach for thermal phase transitions,* Comput. Phys. Commun. **288** (2023) 108725 [2205.08815]

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$$\implies \quad \delta m_3^2 = \frac{\lambda T^2}{4}\,.$$

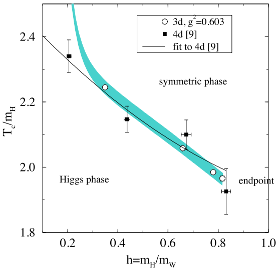
Nowadays automated: DRalgo performs the matching to two loops for a general model.

The second step: integrating out A_0

After the first step the 3d theory still contains the adjoint scalars A_0^a, B_0 with Debye masses $\sim gT$, $\mathcal{L}_{\rm eff} \supset \frac{1}{2} m_{\rm E}^2 A_0^a A_0^a + h_3\, \phi_3^\dagger \phi_3\, A_0^a A_0^a$. Integrating them out shifts

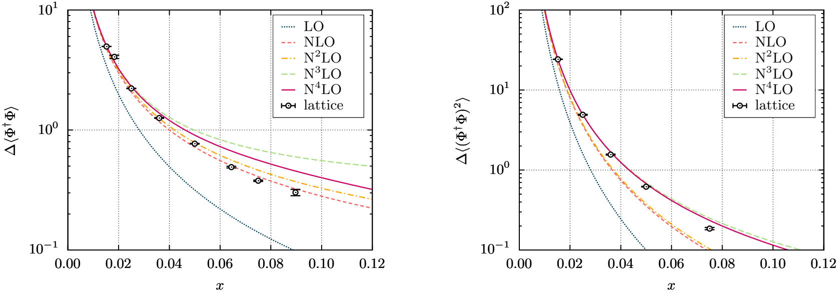
$$\bar{m}_3^2 = m_3^2 - \frac{3h_3}{4\pi} m_{\rm E} - \ldots\,, \qquad \bar{g}_3^2 = g_3^2 \left(1 - \frac{g_3^2}{24\pi m_{\rm E}}\right).$$

The odd powers are the resummed version of the $-m^3 T/12\pi$ term. Careful: this fails when $m_{\rm E} \sim m_{\rm eff}$.

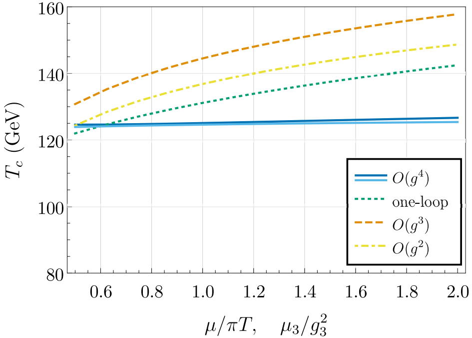


The condensate across the crossover

Gauge-invariant condensates $\langle \phi^\dagger \phi \rangle$ and $\langle (\phi^\dagger \phi)^2 \rangle$ vary smoothly with T: no discontinuity, no latent heat, no gravitational waves.

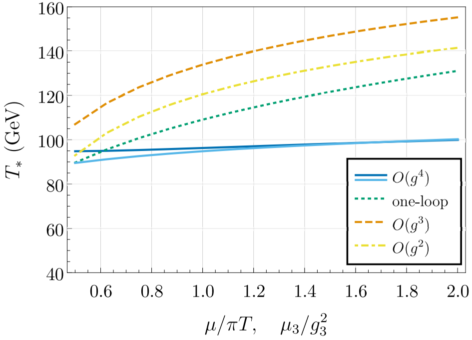


How much supercooling?



At T_{c} the two minima of V_{eff} are degenerate, but nothing happens yet, because $S_{\mathrm{E}} \to \infty$.

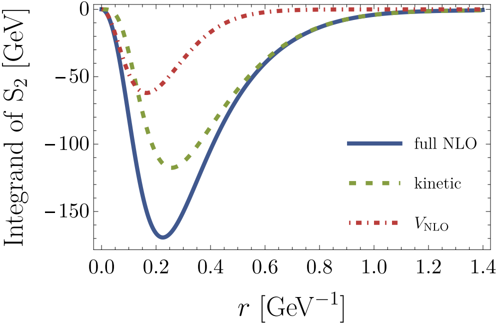
How much supercooling?



By $T_\star$ the universe has supercooled: the barrier has shrunk, S_E has dropped to $\simeq 140$, and bubbles nucleate. How far below T_c this happens is precisely the quantity with the largest theory uncertainty.[1]

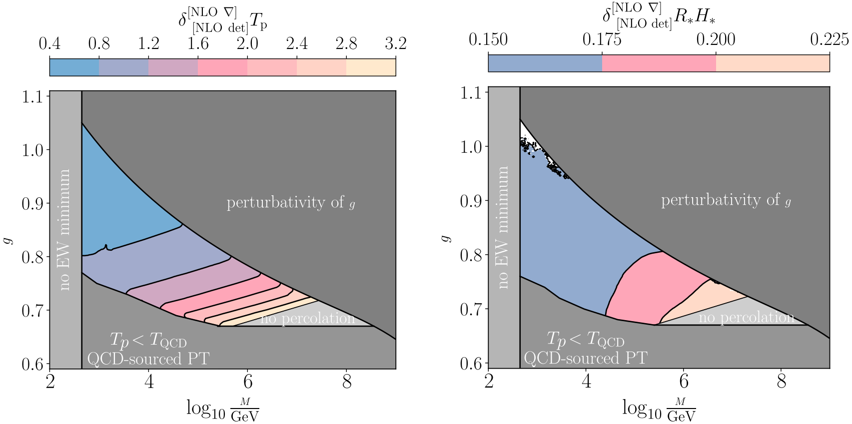
[1] Gould, Tenkanen, *On the perturbative expansion at high temperature and implications for cosmological phase transitions*, JHEP **06** (2021) 069 [2104.04399]

Which contributions matter?



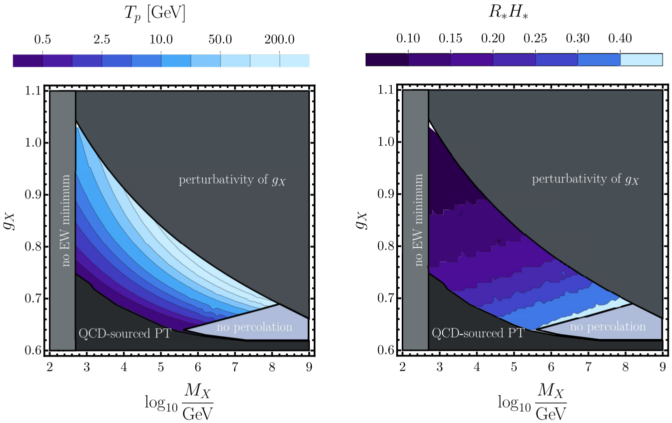
Breakdown of the two-loop contributions to the bounce action: no single diagram dominates, so partial resummations are dangerous.

Gradient expansion vs. full determinant



Relative difference in T_p and in the bubble radius between a gradient expansion of the fluctuation determinant and the full computation. Small where the wall is thin, large where it is thick.

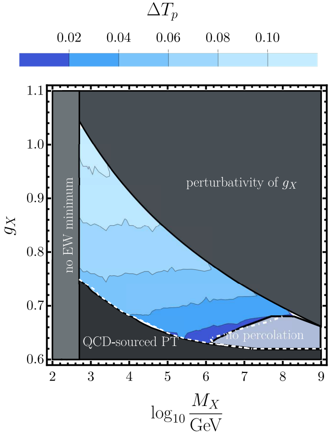
Uncertainty from the renormalization scale



Percolation temperature and bubble radius, computed in the 3d EFT. [1]

[1] Kierkla, Schicho, Swiezewska, Tenkanen, van de Vis, *Finite-temperature bubble nucleation with shifting scale hierarchies,* JHEP **07** (2025) 153 [2503.13597]

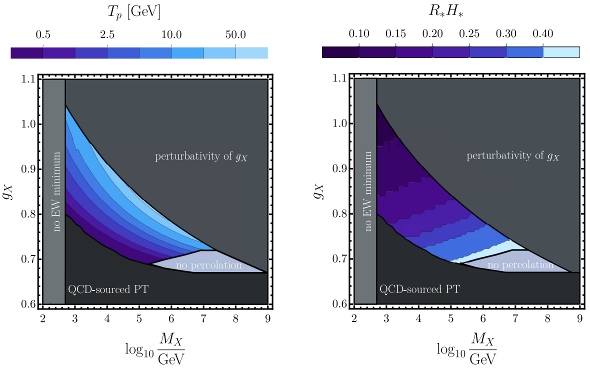
Uncertainty from the renormalization scale



The spread between different renormalization-scale choices.[1]

[1] Kierkla, Schicho, Swiezewska, Tenkanen, van de Vis, *Finite-temperature bubble nucleation with shifting scale hierarchies,* JHEP **07** (2025) 153 [2503.13597]

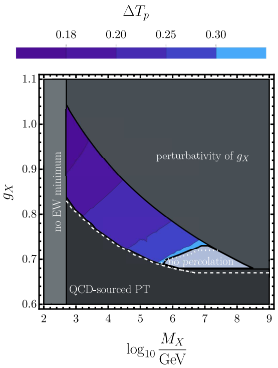
4d vs. 3d: the same quantities, a different organization



The 4d computation has a *much* larger scale dependence: the infrared problem of lecture 1, reappearing in the nucleation rate.

> The uncertainty on T_n is the dominant theory uncertainty in the whole prediction chain.

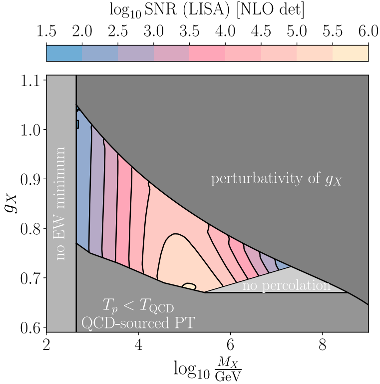
4d vs. 3d: the same quantities, a different organization



The 4d computation has a *much* larger scale dependence: the infrared problem of lecture 1, reappearing in the nucleation rate.

> The uncertainty on T_n is the dominant theory uncertainty in the whole prediction chain.

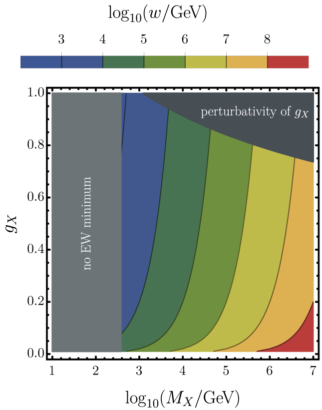
... and what it does to the signal



The signal-to-noise ratio at LISA inherits the nucleation uncertainty: orders of magnitude in Ω_{GW}: the difference between "detectable" and "not detectable".[1]

[1] Croon, Gould, Schicho, Tenkanen, White, *Theoretical uncertainties for cosmological first-order phase transitions,* JHEP **04** (2021) 055 [2009.10080]

Scanning a model



The region with a strong enough transition is narrow, and the theory uncertainty on T_n moves its boundary.

[1] Kajantie, Laine, Rummukainen, Shaposhnikov, *Generic rules for high temperature dimensional reduction and their application to the standard model*, Nucl. Phys. B **458** (1996) 90 [hep-ph/9508379].

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[17] Gould, Tenkanen, *On the perturbative expansion at high temperature and implications for cosmological phase transitions*, JHEP **06** (2021) 069 [2104.04399].

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[18] Kierkla, Schicho, Swiezewska, Tenkanen, van de Vis, *Finite-temperature bubble nucleation with shifting scale hierarchies*, JHEP **07** (2025) 153 [2503.13597].